1. Photon rocket (5 points) — *Solution by Taavet Kalda, grading schemes by ....*

i) *(1 point)* At non-relativistic speeds, we can apply classical momentum and energy conservation to find the acceleration in terms of the antimatter burning rate μ. In a time interval Δt, a mass of $\Delta m = \mu\Delta t$ antimatter annihilates with an equal mass of matter. The resulting photons have an energy equal to the annihilated rest energy $\Delta E = 2\Delta mc^2$. For maximal acceleration, the photons have to be all emitted in the same direction (can be achieved, for example, using mirrors). The resulting photon cloud will then have a momentum of $\Delta p = \Delta E/c$. From the conservation of momentum, the space ship must get a momentum boost in the opposite direction, equal to $\Delta p = M\Delta v = Mg\Delta t$. Combining everything, we get $\mu = Mg/(2c^2) = 5.45\times 10^{-12}\,\mathrm{kg/s}$.

ii) *(3 points)* The final speed is easiest to find by applying energy and momentum (4-momentum) conservation in the initial and final configurations. If the final rest mass of the ship is m_f, then $M - m_f$ of antimatter is burnt throughout the acceleration and an equal amount of matter is burnt from the interstellar space. Hence, it makes sense to consider the system of the ship + the burnt interstellar gas.

The initial rest mass of the system is $M + (M - m_f) = 2M - m_f = M_i$. This corresponds to an energy of $E_i = M_ic^2$ and momentum $p_i = 0$ due to the gas and the ship initially being at rest.

In the final configuration, we have the space ship moving with a speed of v with energy E_f and momentum p_f. We also have a photon cloud with energy E_p and momentum p_p, moving opposite to the direction of the space ship. Then the energy and momentum conservation simply read as $E_f + E_p = E_i$ and $p_i = 0 = p_f + p_p$. We also have the 4-momentum invariant for both the space ship and the photon gas. They read $E_f^2 - p_f^2c^2 = m_f^2c^4$ and $E_p^2 - p_p^2c^2 = 0$. Solving the 4 equations (with 4 unknowns), we get

$$p_f = \frac{M_i^2 - m_f^2}{2M_i}c, \quad E_f = \frac{M_i^2 + m_f^2}{2M_i}c.$$

We could find the velocity by solving for v_f in the expression for the relativistic energy $E_f = m_f\gamma_f c^2$ with $\gamma_f = (1 - v_f^2/c^2)^{-0.5}$. A faster way, however, would be to use the expression for final momentum, $p_f = m_f\gamma_f v$, and notice that

$$\begin{aligned} v_f = \frac{p_f}{E_f}c^2 &= \frac{M_i^2 - m_f^2}{M_i^2 + m_f^2}c \\ &= \frac{(2M - m_f)^2 - m_f^2}{(2M - m_f)^2 + m_f^2}c \\ &= \frac{180}{181}c \approx 0.9945c. \end{aligned}$$

iii) *(1 point)* The last photon is emitted when the space ship moves at a speed of v_f. The photon is observed by a stationary observer. We can directly use the relativistic Doppler shift effect to find

$$\begin{aligned} f_{\text{obs}} = f_0\sqrt{\frac{1 - v/c}{1 + v/c}} &= \frac{m_f}{M_i}f_0 \\ &= \frac{m_f}{2M - m_f}f_0 = \frac{1}{19}f_0. \end{aligned}$$

2. Gas and fluid flows (10 points) — *Solution by Taavet Kalda, grading schemes by ....*

i) *(1 point)* As the plate falls, it will rotate around the bump without slipping and push the air out from beneath it, at ever faster speeds, the closer it gets to the bottom plate. As such, part of the rotational energy of the plate is transferred over to the escaping air molecules. Further, the pressure and temperature of the gas is uniform, because of the incompressibility condition.

Since the problem is 2-dimensional, the mass, volumes, moment of inertia and other quantities are per unit length of the system (on the figure, into the page). Let x mark the distance from the bump and v denote the horizontal speed of air at x. There is a volume of air equal to $V(x) = xhx/(2L)$ between $x = 0$ and x. As the plate falls down, $V(x)$ gets smaller and as a result, air is pushed out. Consider a small time interval $\mathrm{d}t$. In that time interval, h changes by $\dot{h}\mathrm{d}t = -\omega L\mathrm{d}t$. From the conservation of air particles, $0 = \mathrm{d}V(x) + v(x)hx/L$ with $\mathrm{d}V(x) = -x^2\omega/2$. Hence,

$$v(x) = \frac{xL\omega}{2h}.$$

Evaluated at $x = L$, this yields

$$v(x = L) = \frac{L^2\omega}{2h}.$$

ii) *(2.5 points)* Since the air flow is laminar and there is no diffusion, all of the lost rotational energy from the falling glass plate will be converted into kinetic energy of air particles. As such, we have $K_{\text{rot}} + K_{\text{air}} = \text{const}$. We found from the previous parts that the air is pushed out at ever faster speeds (between the plates, $v(x) \gg L\omega$) from between the two plates. However, outside of the two plates, the flow will diffuse fast in all directions, such that the vast majority of kinetic energy will be concentrated between the two plates.

The kinetic energy of the air between the two plates is

$$\begin{aligned} K_{\text{in}} &= \int_0^L \mathrm{d}x\frac{hx}{L}\rho_a\frac{1}{2}v(x)^2 \\ &= \frac{\rho_aL\omega^2}{8h}\int_0^L x^3\mathrm{d}x \\ &= \frac{\rho_aL^5\omega^2}{32h}, \end{aligned}$$

and the rotational energy of the glass plate is

$$K_{\text{rot}} = \frac{I\omega^2}{2} = \frac{m_{\text{glass}}L^2}{3}\frac{\omega^2}{2} = \frac{L^3t\rho_g\omega^2}{6}.$$

Energy conservation then reads as

$$\begin{aligned} \frac{\rho_aL^5\omega^2}{32h} + \frac{L^3t\rho_g\omega^2}{6} &= \text{const} \\ &= \frac{\rho_aL^5\omega_0^2}{32h_0} + \frac{L^3t\rho_g\omega_0^2}{6}, \end{aligned}$$

and hence,

$$\omega = \omega_0\sqrt{\frac{1 + \frac{3\rho_aL^2}{16\rho_gt}\frac{1}{h_0}}{1 + \frac{3\rho_aL^2}{16\rho_gt}\frac{1}{h}}}.$$

We can see that $\lim_{h\to 0}\omega = 0$, i.e. the air acts as a cushion and stops the glass plate before it hits the stationary plate.

iii) *(3 points)* The incoming water from '2' will spread out axisymmetrically along the space between the stone disk and the ceiling. After that, it will spread into the basin and eventually leave through the outgoing pipe '3'. The stone disk is kept up by the pressure differences between the top and bottom of the disc, arising from the flow speed of the water differing on either sides. The fact that pressures differ on boths sides can be seen directly by the application of the Bernoulli Principle $p + \rho_wv^2/2 = \text{const}$ along a streamline or by noting that the flow speed gradients are driven by pressure gradients.

The flow speed outside the gap is negligible due to $t \ll R$, hence we can take the pressure at the bottom side to be uniformly p_0. The flow speed in the gap at a distance x from the axis of symmetry can be found from the conservation of mass applied on a concentric cylinder of radius x and height t, giving $2\pi xt\rho_wv(x) = \mu$. Hence,

$$v(x) = \frac{\mu}{2\pi xt\rho_wv(x)}.$$

Applying Bernoulli's principle, we get $p(x) + \rho_wv^2/2 = p_0$ and so

$$\Delta p = p_0 - p(x) = \frac{\rho_wv^2}{2} = \frac{1}{\rho_w}\left(\frac{\mu}{\pi r}\right)^2.$$

We can hence find the force due to the pressure differences in the gap by integrating from $x = r$ to $x = R$. First note that the resulting force will be pointing vertically up, because $p(x) < p_0$. Integrating,

$$F_1 = \int_{x=r}^{x=R} 2\pi x \mathrm{d}x \Delta p = \frac{\mu^2}{4\pi t^2 \rho_w} \int_{x=r}^{x=R} \frac{\mathrm{d}x}{x} = \frac{\mu^2}{4\pi t^2 \rho_w} \ln\left(\frac{R}{r}\right).$$

Note that the water entering through the pipe will slow down, pushing the disk further down. The net force from this, however, turns out to be negligible due to the condition $r \gg t$. To see this, one can argue that the said force is of order $\mu v_{\text{pipe}} \sim \mu\mu/(\rho_w r^2) \sim \mu^2/(\rho_w r^2) \ll F_1$.

Further, we have the gravitational force $F_g = -mg = -\pi R^2 h \rho_s g$ pulling the disk down. The force balance then reads $F_g + F_1 = 0$. Solving the equation, we find

$$\mu = 2\pi R t \sqrt{\frac{h \rho_w \rho_s}{\ln(R/r)} g}.$$

iv) *(0.5 points)* In the context of thermodynamics, entropy is defined in terms of its differential, such that the change in entropy of a system is given by $\mathrm{d}S = \mathrm{d}Q/T$, where $\mathrm{d}Q$ is the heat entering the system, and T its temperature. Further, entropy in reversible thermodynamic processes is a state function, i.e. it only depends on the current (equilibrium) thermodynamical state of the system. This means that when calculating the entropy difference of one mole of vapour and liquid, the temperature at which the phase transition took place does not affect the final result.

As such, it's most convenient to consider the two final states as only differing by the liquid undergoing condensation at $t_0 = 100\,^\circ\mathrm{C}$. The final temperature is $t_0 = 100\,^\circ\mathrm{C}$ because that's when water vapour pressure is equal to p_0 (i.e. boiling temperature at atmospheric pressure). This corresponds to a heat of $\Delta Q = mL = 1\,\mathrm{mol} \cdot ML$ entering the vapour system, compared to the liquid one. Hence, the entropy difference between one mole of vapour and liquid is given by

$$\Delta S = \frac{\Delta Q}{T_0} = \frac{1\,\mathrm{mol} \cdot ML}{T_0} = 110\,\mathrm{J/K}.$$

v) *(3 points)* Because the expansion of water is reversible, entropy is conserved. This means that the change in entropy due to the expansion of the vapour is balanced by the entropy change due to condensation. As discussed before, because entropy is a function of state, it's most convenient to calculate the entropy change by imagining n moles of water (n will later cancel out) cooling and expanding from T_t, p_t to T_0, p_0 and condensing rn moles of water at the end. r is found by demanding that $\Delta S = 0$ in this process.

The entropy change of the vapour is found by applying the first law of thermodynamics over a small temperature and pressure increment $\mathrm{d}T$, $\mathrm{d}p$:

$$\mathrm{d}S_{\text{vapour}} = \frac{\mathrm{d}Q}{T} = \frac{\mathrm{d}U + \mathrm{d}W}{T},$$

where $\mathrm{d}U = nc_v \mathrm{d}T$ is the change in internal energy of the vapour and $\mathrm{d}W = p\mathrm{d}V$ is the work done by the vapour. Importantly, we neglect the volume of water compared to the vapour, as that allows using the ideal gas to simplify the work differential. Using the ideal gas law, we then have

$$p\mathrm{d}V = p\mathrm{d}\left(\frac{nRT}{p}\right) = nR\mathrm{d}T - nRT\frac{\mathrm{d}p}{p}.$$

Hence,

$$\mathrm{d}S_{\text{vapour}} = n(c_v + R)\frac{\mathrm{d}T}{T} - nR\frac{\mathrm{d}p}{p}$$

and we can integrate to get

$$\Delta S_{\text{vapour}} = nc_p \ln\left(\frac{T_0}{T_1}\right) - nR\ln\left(\frac{p_0}{p_1}\right)$$

where $c_p = c_v + R = R(i+2)/2 = R\gamma/(\gamma-1)$ is the heat capacity at constant pressure. The net change in entropy is then

$$\Delta S_{\text{tot}} = 0 = \Delta S_{\text{vapour}} + \Delta S_{\text{condens}} = \Delta S_{\text{vapour}} - rn\frac{ML}{T_0} = nc_p \ln\left(\frac{T_0}{T_1}\right) - nR\ln\left(\frac{p_0}{p_1}\right) - rn\frac{ML}{T_0}.$$

Thus,

$$r = \frac{RT_0}{ML}\left(\ln\left(\frac{p_1}{p_0}\right) - \frac{\gamma}{\gamma-1}\ln\left(\frac{T_1}{T_0}\right)\right) = 0.122.$$

To find the mass flow rate, we start by finding the flow speed v of the outgoing liquid/vapour. Since we know r, we can apply energy conservation (necessary for the reversibility to hold) on a flowing water packet. This is most conveniently done by demanding energy conservation on the system as a whole. Suppose that in some time interval n moles of water vapour with a volume of V_t are created at the boiler. Then at the outlet, in the steady state, due to conservation of particles, $(1-r)n$ moles of water vapour at a volume of V_0, alongside rn moles of liquid water, are removed. The outflowing water has an additional kinetic energy of $n\mu v^2/2$. The energy change of the whole system due to both steps must cancel each-other out due to energy conservation. We can write this as

$$0 = W_{\text{in}} - W_{\text{out}} + U_{\text{in}} - U_{\text{out}} - nMv^2/2 = 0,$$

where $W_{\text{in}} = p_t V_t = nRT_t$ and $W_{\text{out}} = p_0 V_0 = (1-r)nRT_0$ are the works done by the incoming and outgoing packet and similarly, $U_{\text{in}} = c_v nT_t$ and $U_{\text{out}} = c_v nT_0 - rnML + rnRT_0$ are the internal energies of the incoming and outgoing packet. Notice that for the outgoing internal energy, we have an extra term of $rp_0V_0 = rnRT_0$. This is because when talking about the latent heat of vaporisation, it includes the work done in order to expand the vapour into the volume it's supposed to occupy. Therefore, we should subtract the said work from the latent heat of vaporisation, in order for it to capture the actual change in the internal energy of the water. Combining everything, we get

$$0 = nRT_t - (1-r)nRT_0 + c_v nT_t - c_v nT_0 + rnML - rnRT_0 - nMv^2/2,$$

and hence,

$$v = \sqrt{2\left(\frac{c_p \Delta T}{M} + rL\right)} = 640\,\mathrm{m/s}.$$

The density of air at the outlet is found from ideal gas law $\rho = p_0 M/(RT_0)$. The mass flow rate is then

$$\mu = A\rho v = \frac{Ap_0 Mv}{RT_0} = 37\,\mathrm{g/s}.$$

Grading:V

3. Rotating space station (12 points) — *Solution by Kaarel Hänni, grading schemes by ….*

Grading:

i) *(0.5 points)* Letting ω be the angular velocity, the acceleration experienced by people on the "ground" is $\omega^2 R = \left(\frac{2\pi}{\tau}\right)^2 R$. Setting this equal to g gives

$$\left(\frac{2\pi}{\tau}\right)^2 R = g \implies \tau = 2\pi\sqrt{\frac{R}{g}} \approx 63.437\,\mathrm{s}.$$

ii) *(1.5 points)* Let us consider the motion of the ball in the non-rotating (inertial) frame of the center of mass of the spaceship. As the travel time is $t = \tau/2$, the throwing point on the ground will rotate by exactly half a circle between the ball being thrown and the ball being caught. In our inertial frame, the trajectory of the ball will just be a straight line between these two diametrically opposed points; this has length $2R$. In this inertial frame, the ball thus travels with a constant velocity $v_{\text{inertial}} = \frac{2R}{\tau/2} = \frac{4R}{\tau}$ (in the radial direction). The initial velocity vector in the rotating frame is the difference of the velocity vector in the inertial frame and the velocity vector of the throwing point in the rotating frame compared to the inertial frame. So the initial velocity vector in the rotating frame has a radial component of $\frac{4R}{\tau}$ and a tangential component of $\frac{2\pi R}{\tau}$. The throwing speed s is the magnitude of this vector, which is

$$s = \sqrt{\left(\frac{4R}{\tau}\right)^2 + \left(\frac{2\pi R}{\tau}\right)^2} = \frac{2R}{\tau}\sqrt{4+\pi^2} \approx 117 \text{ m/s}.$$

iii) *(2 points)* When the balloon comes to a stop, it is in equilibrium in the rotating frame. An object at radius R' of mass m_1 has a fictitious radial (downward) force of $m_1\omega^2 R'$ acting on it in this frame. This force on the mass m is $m\omega^2(R-H+l)$. The upward force on the balloon is the difference between this force for the balloon and the buoyant force in the rotating frame, which is $\frac{4}{3}\pi r^3(M-M')\frac{n}{V}\omega^2(R-H) = \frac{4}{3}\pi r^3(M-M')\frac{P}{R_G T}\omega^2(R-H)$. Putting all this together, we can write down the condition that the radial force is 0 in equilibrium

$$m\omega^2(R-H+l) = \frac{4}{3}\pi r^3(M-M')\frac{P}{R_G T}\omega^2(R-H)$$

$$\implies m = \frac{\frac{4}{3}\pi r^3(M-M')P(R-H)}{R_G T(R-H+l)} \approx 112.3 \text{ kg}.$$

iv) *(1.5 points)* In the rotating frame, letting r be the distance from the axis of the cylinder, there is an effective radial potential of

$$\varphi(r) = \int_0^r (-\omega^2 x)\ dx = -\frac{\omega^2 r^2}{2},$$

where we have chosen the potential zero level to be at $r = 0$. In other words, in this frame, there is a fictitious radial force of $m_1\omega^2 r = -m_1\frac{d\varphi}{dr}$ acting on an object of mass m_1. The rope takes a shape that minimizes this potential energy. For this part and the next, we will just be figuring out properties of a rope that minimizes this potential energy (and other than that, we can forget about the rotation). Consider cutting off a tiny piece of rope of length ℓ from point C, then pulling the rope tight at C and gluing it back together (doing work ℓT_C), then cutting the rope open at A, letting it slip by length ℓ (doing work $-\ell T_A$), and finally moving the tiny piece from point C to point A (doing work $\ell\lambda(\varphi(R)-\varphi(R-h)) = -\frac{\ell\lambda\omega^2 h(2R-h)}{2}$), filling the gap of length ℓ. The state of the rope is now the same as initially, so the total work done should be 0:

$$\ell T_C - \ell T_A - \frac{\ell\lambda\omega^2 h(2R-h)}{2} = 0,$$

from where

$$T_A - T_C = -\frac{\lambda 2\pi^2 h(2R-h)}{\tau^2}.$$

See *200 More Puzzling Physics Problems*, problem 78 (and its hint and solution) for a longer explanation of this idea.

v) *(1.5 points)* Let's use the equilibrium condition that torque around the center of the cylinder is 0 (in the rotating frame) for the left half of the rope. Note that the fictitious force is radial, so that contributes nothing to the force. So the only contributions are from the tension in the rope on the two sides, so

$$RT_A\cos\alpha = (R-h)T_C \implies \frac{T_A}{T_C} = \frac{R-h}{R\cos\alpha}.$$

vi) *(1.5 points)* Let the x-axis be the diameter AB, with coordinates chosen to be in meters, and with the coordinates of A being $(-1000,0)$. (So the coordinates of B are $(1000,0)$ and the coordinates of C are $(0,-505)$. The unique parabola $y = ax^2+bx+c$ that goes through these three points has $b = 0$ from symmetry across the y axis, $c = -505$ from condsidering the point C, and then $a = -\frac{505}{10^6}$ from considering the point A. The derivative at A is $\frac{dy}{dx} = 2a\cdot 1000 = -\frac{101}{100}$, from which

$$\cos\alpha = \frac{-dy}{\sqrt{(dy)^2+(dx)^2}} = \frac{1}{\sqrt{1+\left(\frac{dx}{dy}\right)^2}} \approx 0.7106$$

Using the equations from parts (iv) and (v), we now have a system of equations in two unknowns:

$$\begin{cases} T_A - T_C = -\frac{\lambda 2\pi^2 h(2R-h)}{\tau^2} \\ RT_A\cdot 0.7106 = (R-h)T_C \end{cases}.$$

It remains to solve this system of equations. The second equation gives $T_A = T_C\frac{R-h}{R\cdot 0.716} \approx T_C\cdot 0.7053$. Plugging this into the first equation then gives

$$T_C = \frac{\lambda 2\pi^2 h(2R-h)}{\tau^2(1-0.7053)} \approx 12400 \text{ N}.$$

vii) *(1.5 points)* The rotating charge on the walls is making the spaceship into a solenoid carrying current $I = \frac{Q}{\tau}$. Inside of the spaceship, this creates a constant axial magnetic field of magnitude

$$B = \mu_0\frac{I}{L} = \mu_0\frac{Q}{\tau L}.$$

The force this creates on the charged ball is $-qvB = -q\frac{2\pi R}{\tau}\mu_0\frac{Q}{\tau L}$ in the radial upward direction. For the ball to hover above the "ground" motionlessly, the acceleration created by this radial force has to be equal to the centripetal acceleration $\omega^2 R = \frac{(2\pi)^2 R}{\tau^2}$:

$$-\frac{q}{m}\frac{2\pi R}{\tau}\mu_0\frac{Q}{\tau L} = \frac{(2\pi)^2 R}{\tau^2} \implies \frac{q}{m} = -\frac{2\pi L}{\mu_0 Q}.$$

viii) *(1 point)* Consider a charge q_1 at rest at radius r in the rotating frame. The electromagnetic force applied to it is frame-independently $q_1\frac{2\pi r}{\tau}\mu_0\frac{Q}{\tau L}$. This charge is not moving in the rotating frame, so in its frame, the force applied to it by the magnetic field must be 0, and so the electric field measured in its frame must satisfy $q_1 E = q_1\frac{2\pi r}{\tau}\mu_0\frac{Q}{\tau L}$, from where

$$\vec{E} = \frac{2\pi\mu_0 Q}{\tau^2 L}\vec{r}.$$

We now have an expression for E at each point in the rotating frame; what remains is to evaluate $\oint\vec{E}\cdot d\vec{A}$. For the sake of variety, we will demonstrate two ways to evaluate this integral.

For the first option, note that our $\vec{E} = const\cdot\vec{r}$ is exactly the electric field of a uniformly charged cylinder (with the correctly chosen charge density), and so we could equivalently find the same integral around such a uniformly charged cylinder, which Gauss' theorem gives as $V\rho/\epsilon_0 = V\cdot c_1$. We can find an expression for the constant c_1 by considering the simple case where the surface we are dealing with is a cylinder of width 1 m itself, in which case the integral is

$$\oint\vec{E}\cdot d\vec{A} = 1\text{ m}\cdot 2\pi r\frac{2\pi\mu_0 Q}{\tau^2 L}r = V\frac{4\pi\mu_0 Q}{\tau^2 L}.$$

Hence, $c_1 = \frac{4\pi\mu_0 Q}{\tau^2 L}$, and so

$$\oint\vec{E}\cdot d\vec{A} = V\frac{4\pi\mu_0 Q}{\tau^2 L}.$$

To briefly describe a second option for evaluating this integral, note that by partitioning a 3D body into volume slices, with each slice bounded by a volume element $d\vec{A}$ and with cylindrical radius vector $\vec{r}$, and with the volume of each slice being $\vec{r}\cdot d\vec{A}/2$, we get that the volume of the body is the sum of volumes of all such slices,

$$V = \oint\frac{1}{2}\vec{r}\cdot d\vec{A}.$$

This lets us evaluate the main integral as

$$\oint\vec{E}\cdot d\vec{A} = \oint\frac{2\pi\mu_0 Q}{\tau^2 L}\vec{r}\cdot d\vec{A}$$

$$= \frac{4\pi\mu_0 Q}{\tau^2 L}\oint\frac{1}{2}\vec{r}\cdot d\vec{A} = \frac{4\pi\mu_0 Q}{\tau^2 L}V.$$

4. Stretching gloves (8 points) — *Solution by Eero Uustalu, grading schemes by ....*

Part i

Cut a few strips of the material with the same constant width and mark them with two perpendicular lines close to the ends of the strip while leaving ample space for affixing or binding. Then roll one strip lengthwise into a cylindrical shape for stretching. Measure the initial length between the marked lines l_0. With a straight measuring tape affixed to the surface of the table, stretch one of the strips to the breaking point while determining the maximum length between the lines before the strip breaks $l_{\max}$. Repeat this at least three times to see if the data is reproducible. Calculate the result $\epsilon_{\max} = \frac{l_{\max}-l_0}{l_0}$.

In our measurement we got $\epsilon_{\max} = 5.5$

Part ii

We can assume the material has uniform thickness. Knowing that $\sigma = \frac{F}{A}$, $V = const$, and that any force applied to the strip affects all directions perpendicular to the applied force equally, then at any time for the same strip

$$V = a{\cdot}a{\cdot}k{\cdot}l = a_0{\cdot}a_0{\cdot}k{\cdot}l_0 = a_{\max}{\cdot}a_{\max}{\cdot}k{\cdot}l_{\max}$$

where a_0 is the initial thickness of material with no force applied and l_0 is the initial length of the strip with no force applied, a is the thickness when force F is applied and l is the length of strip when force F is applied, $a_{\max}$ and $l_{\max}$ are respective values at breaking force $F_{\max}$.

k is the ratio between the width d and the thickness a of the strip such that $d = k{\cdot}a$ where k remains constant for the strip while stretching. Therefore

$$\sigma = \frac{F}{a{\cdot}a{\cdot}k} \text{ and } \sigma_{\max} = \frac{F_{\max}}{a_{\max}{\cdot}a_{\max}{\cdot}k}$$

From $V = const$ we can deduce that $a{\cdot}a = \frac{a_0{\cdot}a_0{\cdot}l_0}{l}$ and $a_{\max}{\cdot}a_{\max} = \frac{a_0{\cdot}a_0{\cdot}l_0}{l_{\max}}$

If we could apply the same force F on strips of different initial widths $d = k{\cdot}a$ we could vary the tension for each strip. Therefore we would be using different strips with different k values for each measurement

$\sigma = \frac{F}{k}\cdot\frac{l}{l_0}\cdot\frac{1}{a_0{\cdot}a_0}$ and $\sigma_{\max} = \frac{F}{k_{\max}}\cdot\frac{l_{\max}}{l_0}\cdot\frac{1}{a_0{\cdot}a_0}$

Where $k = d/a$ and $k_{\max} = d_{\max}/a_{\max}$ are the width to thickness rations of the the respective strips used for F and $F_{\max}$. By combining the above, we get

$$\frac{\sigma}{\sigma_{\max}} = \frac{k_{\max}}{k}\cdot\frac{l}{l_{\max}}$$

but since $d_0 = k{\cdot}a_0$ is the initial unstressed width of one strip and $d_{\max 0} = k_{\max}{\cdot}a_0$ is the initial unstressed width of another strip stretched to the breaking point. For building graph it transforms to:

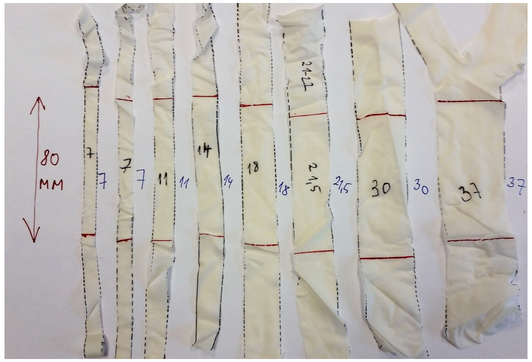
$\frac{\sigma}{\sigma_{\max}} = \frac{d_{\max 0}}{d_0}\cdot\frac{l}{l_{\max}}$ for the Y axis and

$\epsilon = \frac{l-l_0}{l_0}$ for the X axis.

Measurements

We can assume the material has uniform thickness. We want to apply the same force on strips with different initial widths.

Solution 1



We make many strips with different widths, mark them with the same initial length l_0 using perpendicular lines, measure and record the initial widths d_0 of each segment, then roll them lengthwise into a cylindrical shape and bind them one after each another. The force will be the same for all segments but the value of σ for each segment will depend on the initial width of each individual strip.

A range where $d_{\min}$ and $d_{\max}$ differ at least 8 times is recommended.

Rolling the strip lengthwise into a cylindrical shape before binding gives the possibility to distribute the force evenly on all the strips regardless of the width of the strip!

Our solution was to make two identical strips with the smallest possible width and attach them to one another at one end to form a Y shape. The previously determined $\epsilon_{\max}$ was used to estimate the possible maximum stretch almost to the breaking point of the narrowest strip. In case of a breakage event the identical spare strip at the other end of the Y shape could be used for measurement. After affixing the multi-width rope of combined strips at its maximum stretch position, the values of l were recorded for each individual section. Afterwards he rope was stretched till break to find the actual breaking length $l_{\max}$ of the narrowest strip.

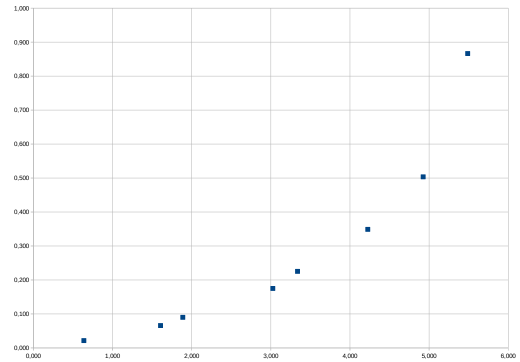
The $7mm$ wide initially $80mm$ long test strip break at length $522mm$. Therefore the measurement of the rope of strips was made using force F stretching the second $7mm$ wide $80mm$ test strip to $519mm$.

But the $7mm$ stripe measured break when it was stretched to $599mm$.

So the $d_{\max 0} = 7mm$ and $l_{\max} = 599mm$.

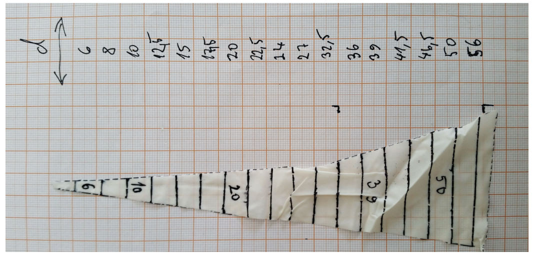
$(l-l_0)/l_o$	mm l_0	mm l	mm d_0	$\sigma/\sigma_{\max}$
0,638	80	131	71	0,022
1,606	80	208,5	37	0,066
1,888	80	231	30	0,090
3,025	80	322	21,5	0,175
3,338	80	347	18	0,226
4,225	80	418	14	0,349
4,925	80	474	11	0,505
5,488	80	519	7	0,866

And the graph, $\frac{\sigma}{\sigma_{\max}}$ for the Y axis and ϵ for the X axis.

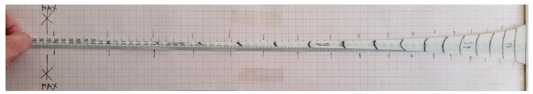


Solution 2

We make two identical elongated triangles from the longest piece of material available. We mark the triangles with evenly spaced perpendicular lines. (The lines are perpendicular to the central symmetry line of the elongated triangle.)



We record the initial length l_0 and the average width d_0 of each segment as defined by the perpendicular lines. When stretched, all segments will have the same force applied but each segment will have its own σ. We have to affix the base of the prolonged triangle to the ruler with tape (it has to be a good strong joint) and we affix the ruler to the table. A long strip of graph paper was taped to the table to have the rubber triangle stretched over it.



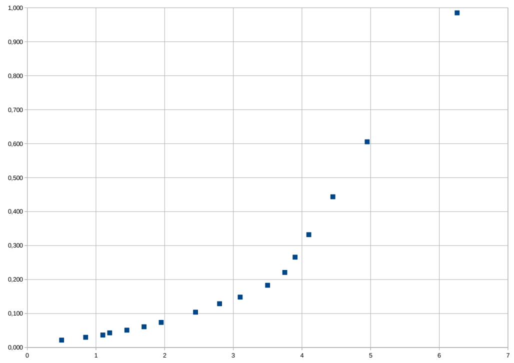
The triangle was stretched until the narrow end segment was near breaking point, and the values of l were recorded for each segment. (Most easily done by using a marker on the graph paper while holding the narrow end of the triangle at a fixed position with the other hand). One of the triangles was actually stretched to the braking point, the other triangle was stretched to the breaking point only after the measurement was made.

The narrow end segment of the first triangle of $10mm$ long and the average width $6mm$ break at length $73{,}1mm$. Therefore the measurement of the second similar triangle tested was made using force $\boldsymbol{F}$ stretching the narrow end segment of the triangle to $72{,}6mm$. The actual break of narrow segment of the second triangle occurred at length $73{,}7mm$.

So the $d_{\max 0} = 6mm$ and $l_{\max} = 73{,}7mm$.

$\sigma/\sigma_{\max}$	mm l_0	mm l	mm d_0	$(l-l_0)/l_o$
0,985	6	10	72,6	6,26
0,605	8	10	59,5	4,95
0,444	10	10	54,5	4,45
0,333	12,5	10	51	4,1
0,266	15	10	49	3,9
0,221	17,5	10	47,5	3,75
0,183	20	10	45	3,5
0,148	22,5	10	41	3,1
0,129	24	10	38	2,8
0,104	27	10	34,5	2,45
0,074	32,5	10	29,5	1,95
0,061	36	10	27	1,7
0,051	39	10	24,5	1,45
0,043	41,5	10	22	1,2
0,037	46,5	10	21	1,1
0,030	50	10	18,5	0,85
0,022	56	10	15	0,5

And the graph, $\frac{\sigma}{\sigma_{\max}}$ for the $\boldsymbol{Y}$ axis and ϵ for the $\boldsymbol{X}$ axis.



Grading: